Time-averaging depth for metabolic rate calculations

Derivation (1): conservation of mass

Metabolic rates are calculated from the integrated changes of TA and DIC as water parcels traverse the reef flat. The equations for net ecosystem calcification (NEC) and productivity (NEP) are derived from mass balance of TA and DIC. Here, we derive the Lagrangian NEC equation:

$$\frac{-d\mathrm{TA}}{dt}\frac{1\ \mathrm{mol\ CaCO_3}}{2\ \mathrm{eq\ TA}}\rho h - \mathrm{NEC} = 0 \qquad (1)$$

where $\frac{d\mathrm{TA}}{dt}$ is the instantaneous rate of change of TA (eq kg^{-1}) with respect to time *t* (hr), ρ is seawater density (kg m^{-3}), *h* is depth (m), NEC is the benthic $CaCO_3$ flux (mol C m^{-2} hr^{-1}) and 2 equivalents of alkalinity are required to form 1 mole of $CaCO_3$. We assume that NEC is the only process affecting seawater TA. Rearranging Eq. (1):

$$\frac{-d\mathrm{TA}}{dt}\rho h = 2\mathrm{NEC} \qquad (2).$$

Within a Lagrangian framework, we track a given parcel (mass) of seawater. For simplicity, we can define the term ta as the total equivalents of alkalinity in a given mass of seawater that we will trace across the reef:

$$\mathrm{ta} = m\mathrm{TA} \qquad (3)$$

where *m* is mass (kg) and thus:

$$\frac{-d\mathrm{ta}}{dt}\frac{\rho h}{m} = 2\mathrm{NEC} \qquad (4).$$

The benthic area of reef interacting with our mass of seawater is (see also Figure 1):

$$A = \frac{m}{\rho h} \qquad (5)$$

where *A* is area (m^2). Substituting Eq. (5) into Eq. (4):

$$\frac{-d\mathrm{ta}}{dt}\frac{1}{A} = 2\mathrm{NEC} \qquad (6).$$

Eq. (6) represents the instantaneous rate of change of alkalinity in our mass of seawater per area of reef. In practice, we measure the integrated change of alkalinity over a given residence time on the reef:

$$\overline{\mathrm{NEC}} = \frac{1}{2n}\sum_{i=1}^{n}\frac{\mathrm{ta}_{i-1}-\mathrm{ta}_i}{t_i-t_{i-1}}\frac{1}{A_i} \qquad (7)^1$$

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[1]In integral form: $\overline{\mathrm{NEC}} = \frac{1}{2\tau}\int_0^{\tau}\frac{\partial \mathrm{ta}}{\partial t}\frac{1}{A(t)}dt$, which gives the same answer in the end.

where the summation is evaluated from the *i*th to the *n*th total time points, and $\overline{\text{NEC}}$ is the average NEC rate. Conceptually, there are *n* different instantaneous rates across the reef flat, and the mean of these rates with respect to time produces the measured alkalinity signal. At each time point *i*, the rate of the accumulation of the alkalinity signal depends on the area of reef interacting with the mass of seawater. For example, signals accumulate faster in shallower water because the mass of seawater is spread across a greater benthic area (Figure 1). Dividing alkalinity change by area thus normalizes the NEC rate. Since we measure alkalinity at only 2 time points (offshore source-water and on the reef flat), we can simplify Eq. (7) as:

$$\overline{\text{NEC}} = \frac{1}{2n}\sum_{i=1}^{n}\frac{\Delta\text{ta}}{\tau}\frac{1}{A_i} = \frac{\Delta\text{ta}}{2\tau}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{A_i} \tag{8}$$ [2]

where Δta is the total (measured) alkalinity change ($\sum_{i=1}^{n}\text{ta}_{i-1} - \text{ta}_i$) and τ is the residence time of the mass of seawater on the reef ($\sum_{i=1}^{n} t_i - t_{i-1}$). In Eq. (8), we are now averaging only the benthic area interacting with our seawater mass (*A*) across time points. Substituting Eq. (5) into Eq. (8):

$$\overline{\text{NEC}} = \frac{\Delta\text{ta}}{2\tau}\frac{1}{n}\sum_{i=1}^{n}\frac{\rho_i h_i}{m_i} \tag{9}$$

where ρ_i is the seawater density at time *i*, h_i is the local depth at time *i* when the seawater mass is in a certain location on the reef, and m_i is the mass of the seawater parcel, which we have set as constant in our Lagrangian framework (*i.e.* mass at all times $m_i = m$) (Figure 1). Seawater density changes by very small percentages depending on temperature and salinity, and thus we can assume a constant seawater density across the reef flat (*i.e.* density at all times $\rho_i = \rho$). Therefore, Eq. (9) simplifies to:

$$\overline{\text{NEC}} = \frac{\Delta\text{ta}}{2\tau}\frac{\rho}{m}\frac{1}{n}\sum_{i=1}^{n} h_i \tag{10}$$ [3]

where it can be seen that we need to average with respect to time only depth as the seawater mass traverses the reef flat. Substituting Eq. (3) back into Eq. (10):

$$\overline{\text{NEC}} = \frac{\Delta\text{TA}}{2\tau}\rho\frac{1}{n}\sum_{i=1}^{n} h_i \tag{11}$$

we now express the equation with the measured TA change (eq kg^{-1}) between offshore and reef water. Defining the mean depth with respect to time:

$$\bar{h} = \frac{1}{n}\sum_{i=1}^{n} h_i \tag{12}$$

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[2] Since we only have 2 TA measurements, we simplify this step as $\overline{\text{NEC}} = \frac{1}{2n}\left(\frac{\Delta\text{ta}}{\tau}\frac{1}{A_i} + \frac{\Delta\text{ta}}{\tau}\frac{1}{A_{i+1}} + \cdots \frac{\Delta\text{ta}}{\tau}\frac{1}{A_n}\right) = \frac{\Delta\text{ta}}{\tau}\frac{1}{2n}\left(\frac{1}{A_i} + \frac{1}{A_{i+1}} + \cdots \frac{1}{A_n}\right)$.

[3] Equivalent to $\overline{\text{NEC}} = \frac{\Delta\text{ta}}{2\tau}\frac{1}{n}\left(\frac{\rho h_i}{m} + \frac{\rho h_{i+1}}{m} + \cdots \frac{\rho h_n}{m}\right) = \frac{\Delta\text{ta}}{2\tau}\frac{\rho}{m}\frac{1}{n}(h_i + h_{i+1} + \cdots h_n)$

and substituting Eq. (12) into Eq. (11) we can define the simplified NEC equation:

$$\overline{\mathrm{NEC}} = \frac{\Delta \mathrm{TA}}{2\tau}\rho\bar{h} \quad (13)$$

and assuming we only measured TA of source-water and on the reef flat (rf):

$$\overline{\mathrm{NEC}} = \frac{\mathrm{TA_{offshore}} - \mathrm{TA_{rf}}}{2\tau}\rho\bar{h} \quad (14).$$

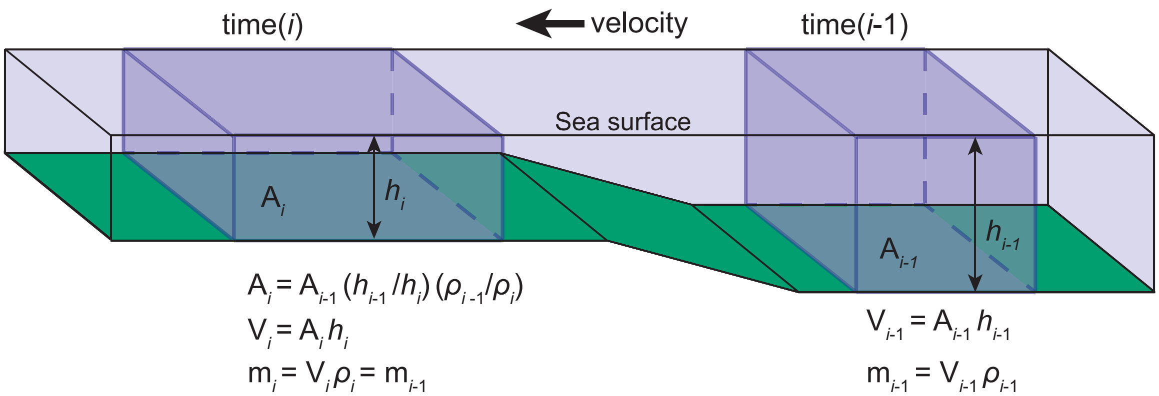

Figure 1. Conceptual diagram showing conservation of mass for tracing a parcel of seawater across the reef flat. The same water parcel is shown at only two different times (*i.e.* different locations if velocity > 0) for simplicity. Dimensions of the water parcel change with variability in bathymetry and/or density. At both time points, the parcel has the same mass, but covers a different benthic area.

Derivation (2): equal benthic area

The Lagrangian NEC equation can also be considered by tracking alkalinity changes across the reef in the water column above a constant benthic area. While the general approach is the same as the first derivation above, this equal benthic area approach will not keep a constant mass of seawater. Beginning with Eq. (2) above:

$$\frac{-d\mathrm{TA}}{dt}\rho h = 2\mathrm{NEC} \quad (2)$$

and expressing the mean NEC as the mean of TA changes in time across the reef above a constant benthic area:

$$\overline{\mathrm{NEC}} = \frac{1}{2n}\sum_{i=1}^{n}\frac{\mathrm{TA}_{i-1} - \mathrm{TA}_i}{t_i - t_{i-1}}\rho_i h_i \quad (15).$$

where we are tracking TA (eq kg^{-1}) changes in the water column above a certain area. As in the first derivation above, there are *n* different instantaneous rates across the reef flat, and the mean of these rates with respect to time produces the measured TA signal. At each time point *i*, the rate of the accumulation of the alkalinity signal depends on the depth of the water column. For example, signals accumulate faster in shallower water because the benthic NEC flux is imposed on a lesser mass of seawater (Figure 2). Multiplying TA change by depth thus normalizes the NEC rate. Since we measure TA at only 2 time points (offshore source-water and the reef flat), we can simplify Eq. (15) as:

$$\overline{\mathrm{NEC}} = \frac{\Delta \mathrm{TA}}{2\tau}\frac{1}{n}\sum_{i=1}^{n} \rho_i h_i \qquad (16).$$

Seawater density changes by very small percentages depending on temperature and salinity, and thus we can assume a constant seawater density across the reef flat (*i.e.* density at all times $\rho_i = \rho$). Therefore, Eq. (16) simplifies to:

$$\overline{\mathrm{NEC}} = \frac{\Delta \mathrm{TA}}{2\tau}\rho\frac{1}{n}\sum_{i=1}^{n} h_i \qquad (17)$$

which is identical to Eq. (11) above, and thus following Eqs. (12-14) we arrive at the same final NEC equation as the first derivation:

$$\overline{\mathrm{NEC}} = \frac{\mathrm{TA_{offshore}} - \mathrm{TA_{rf}}}{2\tau}\rho\bar{h}$$

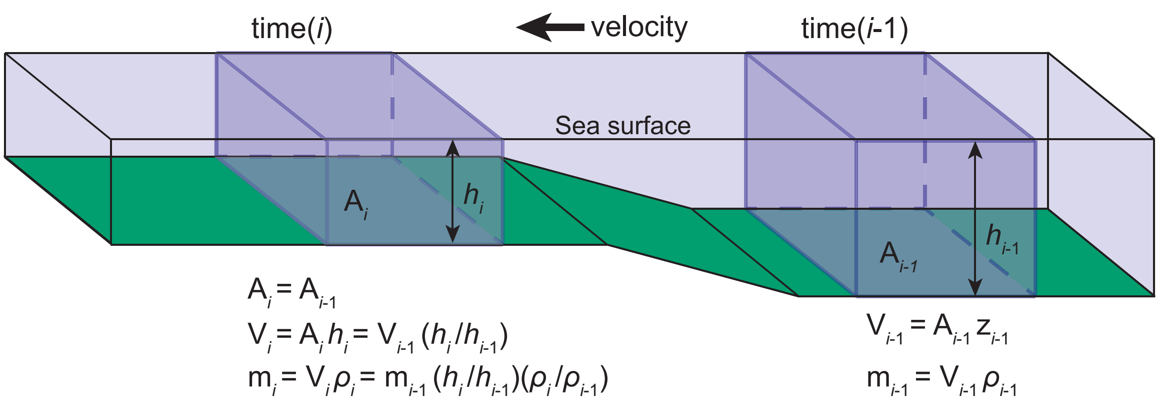

Figure 2. Conceptual diagram showing changes in mass of the water column as a water parcel moves across variable bathymetry. At both time points, the parcels have the same benthic areas, but different masses.

Tracing of seawater parcels to estimate residence time and time-average depth:

Our estimates of metabolic rates (NEC and NEP) are based on a quasi-Lagrangian framework (*i.e.* we track water parcels indirectly with current velocities and bathymetry profiles, rather than directly with dye tracers or drogues). For each water sample collected on the reef flat, residence time (τ) of the of the sampled seawater parcel can be estimated by back-tracking in time from the reef flat sampling location to the reef crest using depth-averaged cross-shore current velocity (**u**) measurements, local bathymetry across the reef flat (h), and tidal changes in sea level. We assume that transport ($q = \mathbf{u}h$) and flow direction are conserved across the reef flat (*i.e.* $\mathbf{u}h(t)$ = constant at all locations, x, across the reef flat). The distance x along the reef flat (*i.e.* distance from the reef crest) of a water parcel is therefore estimated at any time t by:

$$\mathrm{x}(t) = \mathrm{x}(0) - \int_0^t \frac{\mathbf{u}_{\mathrm{rf}}(t) \cdot h_{\mathrm{rf}}(t)}{h(t,\mathrm{x})} dt \tag{21}$$

where x is distance in meters from the reef crest, t is time in seconds, x(0) is the final position (m) at the time of sampling (*i.e.* distance from reef flat sampling site to the reef crest), $\mathbf{u}_{\mathrm{rf}}$ is the depth-averaged current velocity (m s^{-1}) measured by the ADP on the reef flat, h_{rf} is depth at reef flat sampling site, and $h(t,\mathrm{x})$ is the local depth at time t and distance d along the path of the water parcel as it traverses the reef flat. Integrating Eq. 21 backward in time until x = 0, we calculate the residence time as the difference between the sampling time at the reef flat site and the time when the water parcel first encountered the reef at the reef crest. The time-averaged depth of a water parcel as it traverses the reef flat is thus:

$$\bar{h} = \frac{1}{\mathrm{n}} \sum_1^{\mathrm{n}} h(t, \mathrm{x}) \tag{22}$$

where n is the number of discrete, evenly-spaced points in time at which a water parcel was traced (*i.e.* the sampling times of the ADP).